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Time Series Analysis

Asymptotic Results for Spatial ARMA Models

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Causal quadrantal-type spatial ARMA(p, q) models with independent and identically distributed innovations are considered. In order to select the orders (p, q) of these models and estimate their autoregressive parameters, estimators of the autoregressive coefficients, derived from the extended Yule–Walker equations are defined. Consistency and asymptotic normality are obtained for these estimators. Then, spatial ARMA model identification is considered and simulation study is given.

Keywords Asymptotic properties; Causality; Estimation; Order selection; Spatial autoregressive moving-average models.

Mathematics Subject Classification Primary 62M30, 62M10; Secondary 60F05.

1. Introduction

Unlike the time series autoregressive moving-average (ARMA) case, several kind of spatial ARMA models may be defined (cf. Guyon, 1995, Tjøstheim, 1978, 1983). Most of the different ARMA representations appear to depend on the order chosen on the lattice $\mathbb{Z}^d$.

For spatial autoregressive (AR) model having quadrantal representation, Tjøstheim (1983) considered Yule–Walker and least squares (LS) estimators for the parameters and proved the strong consistency of both these estimators and the asymptotic normality for the LS estimator, this even when the innovations of these AR models are strong martingale-differences. Ha and Newton (1993) proved that the Yule–Walker estimator considered in Tjøstheim (1983) is in fact biased for the asymptotic normality. They gave the explicit expression of this bias for a causal AR random field indexed by a two-dimensional lattice and proposed an new estimator called "unbiased Yule–Walker estimator" that has the asymptotically normal property.

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Tjøstheim (1983) used a central limit theorem for strong martingale-differences, relative to the partial ordering on $\mathbb{Z}^d$, to obtain the asymptotic normality of the LS estimators. For weaker martingale-differences, defined with the lexicographic order, Huang (1992) also obtained a central limit theorem. While using this total ordering, non symmetric half-space ARMA models can be defined. Huang and Anh (1992) considered such models on a two-dimensional lattice. They gave a method based on an inverse model associated with the original one to estimate the orders and the parameters of the model avoiding in this way the estimation of the innovations.

Other kind of spatial ARMA process, not depending on the order on $\mathbb{Z}^d$, can also be defined. Etchison et al. (1994) introduced two-dimensional lattice separable ARMA models, which are in fact "products" of two ARMA processes indexed over one-dimensional lattice set. They studied the properties of the sample partial autocorrelation function in order to identify orders in AR models. Furthermore, Shitan and Brockwell (1995) proposed an asymptotic test of separability using the properties of the autocovariance function of a spatial separable AR model.

In this article, we are interested in model order selection and in AR parameters estimation of a causal quadrantal ARMA random fields as defined in Tjøstheim (1978) with independent and identically distributed innovations. In this purpose, we introduce estimators of the AR coefficients based on a derivation of extended Yule–Walker equations. These estimators correspond in fact to an extension to spatial case of the sample generalized partial autocorrelation (GPAC) function known in time series (Woodward and Gray, 1981). Unlike the one-dimensional lattice case, several estimators of the GPAC function, that are unbiased for asymptotic normality, could be defined. Our estimators differ from those retained in Ha and Newton (1993). We establish that they are consistent and asymptotically normal. For this, we prove in the first sections some asymptotic properties of sample autocovariances and sample crosscovariances of two linear random fields. These results extend those in Choi (2000) for moving-average (MA) random fields and also generalize the known corresponding results for time series (Brockwell and Davis, 1987). In the last section, we apply some obtained asymptotic results when considering the order selection of a spatial ARMA((1, 1), (1, 1)) model and the estimation of its AR parameters. A simulation study is also given.

2. Preliminaries and Assumptions

In this section, we recall some main notations on random fields indexed over $\mathbb{Z}^d$. In all the following, all the random fields are indexed over $\mathbb{Z}^d$, with $d \geq 2$, and $\mathbb{Z}^d$ is endowed with the usual partial order that is for $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$ in $\mathbb{Z}^d$, we write $s \leq t$ if for all $i = 1 \ldots d$, $s_i \leq t_i$. For $a, b \in \mathbb{Z}^d$, such that $a \leq b$ and $a \neq b$, the following indexing subsets in $\mathbb{Z}^d$, will be considered:

$$S[a, b] = \{x \in \mathbb{Z}^d \mid a \leq x \leq b\}, \quad S\langle a, b] = S[a, b] \backslash \{a\},$$
$$S[a, \infty] = \{x \in \mathbb{Z}^d \mid a \leq x\}, \quad S\langle a, \infty] = S[a, \infty] \backslash \{a\}.$$

Let $(X_t)_{t \in \mathbb{Z}^d}$ be a real valued square integrable random field. Its autocovariance function γ is defined by $\gamma(u, v) = \mathbb{E}[(X_u - \mathbb{E}(X_u))(X_v - \mathbb{E}(X_v))]$ for u and v in $\mathbb{Z}^d$.

The random field (X_t) is said to be stationary if

$$\mathbb{E}(X_t) = m \quad \forall t \in \mathbb{Z}^d,$$
$$\gamma(u, v) = \gamma(u + h, v + h) \quad \forall u, v, h \in \mathbb{Z}^d$$

and strictly stationary if for all $h, a, b \in \mathbb{Z}^d$, $(X_j)_{j \in S[a,b]}$ and $(X_{j+h})_{j \in S[a,b]}$ have the same joint distributions.

As $\gamma(u, v) = \gamma(u - v, 0)$ for all $u, v \in \mathbb{Z}^d$ when $(X_t)_{t \in \mathbb{Z}^d}$ is a stationary random field, it is convenient to redefine the autocovariance function as a function of one argument as follows:

$$\gamma(h) = \gamma(h, 0).$$

Throughout this work, $(\epsilon_t)_{t \in \mathbb{Z}^d}$ denotes a family of independent and identically distributed (i.i.d) centered random variables with variance $\sigma^2 > 0$. We say that a random field $(X_t)_{t \in \mathbb{Z}^d}$ is linear if for any t,

$$X_t = \sum_{j \in \mathbb{Z}^d} \psi_j \epsilon_{t-j} \tag{1}$$

with $\sum |\psi_j| < \infty$.

Remark 2.1. Linear random fields as defined above are strictly stationary. They are called MA random fields in Choi (2000).

For two linear random fields $(X_t)_{t \in \mathbb{Z}^d}$ and $(Y_t)_{t \in \mathbb{Z}^d}$, we define their crosscovariance function as $\gamma_{xy}(h) = \mathbb{E}(X_t Y_{t+h})$ for every h in $\mathbb{Z}^d$.

Given two samples $\{X_t, t \in S[\mathbf{1}, \mathbf{N}]\}$ and $\{Y_t, t \in S[\mathbf{1}, \mathbf{N}]\}$, we define the sample crosscovariance function as follows:

$$\hat{\gamma}_{xy}(h) = \frac{1}{N_h} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t+h \in S[\mathbf{1},\mathbf{N}]}} X_t Y_{t+h} \tag{2}$$

with $N_h = \prod_{i=1\ldots d}(N - h_i)$ and where for $N \in \mathbb{N}\backslash\{0\}$, we denote by $\mathbf{N}$ the element of $\mathbb{Z}^d$ whose all components equal N.

In the particular case when $X_t = Y_t$, γ and $\hat{\gamma}$ are, respectively denoted for γ_{xy} and $\hat{\gamma}_{xy}$.

Following Tjøstheim (1978) and Guyon (1995), we say that a random field $(X_t)_{t \in \mathbb{Z}^d}$ is a spatial ARMA(p, q) with parameters $p, q \in \mathbb{Z}^d$ if it is stationary and satisfies the following equation

$$X_t - \sum_{j \in S\langle 0,p]} \phi_j X_{t-j} = \epsilon_t + \sum_{k \in S\langle 0,q]} \theta_k \epsilon_{t-k} \tag{3}$$

where $(\phi_j)_{j \in S\langle 0,p]}$ and $(\theta_k)_{k \in S\langle 0,q]}$ denotes, respectively the autoregressive and the moving-average parameters with the convention that $\phi_0 = \theta_0 = 1$. If p (respectively, q) equals 0, the sum over $S\langle 0, p]$ (respectively, $S\langle 0, q]$) is supposed to be 0 and the process is called an AR(p) (respectively, MA(q)) random field.

The ARMA random field is called causal if it has the following unilateral expansion

$$X_t = \sum_{j \in S[0,\infty]} \psi_j \epsilon_{t-j}, \tag{4}$$

with $\sum |\psi_j| < \infty$.

Remark 2.2. Let $\phi(z) = 1 - \sum_{j \in S\langle 0,p]} \phi_j z^j$ and $\theta(z) = 1 + \sum_{j \in S\langle 0,q]} \theta_j z^j$ where $z = (z_1, \ldots, z_d)$. Then, a sufficient condition (c.f. Tjøstheim, 1978) for the random field to be causal is that the autoregressive polynomial $\phi(z)$ has no zeroes in the closure of the open disc D^d in $\mathbb{C}^d$.

For the identifiability of model (3), we assume in addition that these two polynomials have no common irreductible factors in the factorial ring $\mathbb{C}[z_1, \ldots, z_d]$.

The ARMA order selection procedure that we consider herein is based on extended Yule–Walker equations. More precisely, for spatial ARMA(p, q) random field $(X_t)_{t \in \mathbb{Z}^d}$ with $p \neq 0$, we define for $\lambda \in S\langle 0, \infty]$ and $v \in S[0, \infty]$, the coefficients $\phi_\lambda^{(v)} = (\phi_{\lambda,j}^{(v)})_{j \in S\langle 0,\lambda]}$ as the solution, when it exists, of the following extended Yule–Walker equations,

$$\mathbb{E}(Y_{\lambda,t}^{(v)} X_{t-v-j}) = 0 \quad \forall j \in S\langle 0, \lambda], \tag{5}$$

where $Y_{\lambda,t}^{(v)} = X_t - \sum_{j \in S\langle 0,\lambda]} \phi_{\lambda,j}^{(v)} X_{t-j}$.

If we arrange vectors and matrix in the lexicographic order, these equations can be rewritten under the form of the following single matricial equation

$$\Gamma_\lambda^{(v)} \phi_\lambda^{(v)} = \gamma_\lambda^{(v)} \tag{6}$$

with for all $i, j \in S\langle 0, \lambda]$

$$\Gamma_\lambda^{(v)}(j, i) = \gamma(v + j - i),$$

and

$$\gamma_\lambda^{(v)}(j) = \gamma(v + j).$$

Given a sample $\{X_t, t \in S[\mathbf{1}, \mathbf{N}]\}$, we could have considered the extended Yule–Walker estimator $\bar{\phi}_\lambda^{(v)}$ of $\phi_\lambda^{(v)}$ defined as the solution of the following matricial equation

$$\overline{\Gamma}_\lambda^{(v)} \bar{\phi}_\lambda^{(v)} = \bar{\gamma}_\lambda^{(v)},$$

where for $i, j \in S\langle 0, \lambda]$,

$$\overline{\Gamma}_\lambda^{(v)}(j, i) = \overline{R}(v + j - i),$$
$$\bar{\gamma}_\lambda^{(v)}(j) = \overline{R}(v + j),$$

with for $h \in S\langle 0, \tau]$,

$$\overline{R}(h) = \frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-h \in S[\mathbf{1},\mathbf{N}]}} X_t X_{t-h}.$$

As mentioned above, Ha and Newton (1993) have proved for an AR(p) random field, indexed over $\mathbb{Z}^2$, that the estimator $\bar{\phi}_{\lambda}^{(v)}$ is biased for the asymptotic normality. The bias appears because, in the terms $\overline{R}(h)$, the summation set $\{t \in S[\mathbf{1}, \mathbf{N}], t-h \in S[\mathbf{1}, \mathbf{N}]\}$ depends on h and simultaneously, the normalization coefficient N^d does not match to its cardinal. By modifying the normalization coefficient, they proposed then an unbiased estimator called "unbiased Yule–Walker estimator" for which they proved the asymptotic normality. Here we take another approach to obtain unbiased estimators from the extended Yule–Walker equations (5). More precisely, we define the estimators $\hat{\phi}_{\lambda}^{(v)} = (\hat{\phi}_{\lambda,j}^{(v)})_{j \in S\langle 0,\lambda]}$ of the coefficients $(\phi_{\lambda,j}^{(v)})_{j \in S\langle 0,\lambda]}$ by the following equations

$$\frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} \widehat{Y}_{\lambda,t}^{(v)} X_{t-v-j} = 0 \quad \forall\, j \in S\langle 0, \lambda] \tag{7}$$

where $\widehat{Y}_{\lambda,t}^{(v)} = X_t - \sum_{j \in S\langle 0,\lambda]} \hat{\phi}_{\lambda,j}^{(v)} X_{t-j}$ and $\tau = \lambda + v$.

These equations can also be rewritten under the following matricial form

$$\widehat{\Gamma}_{\lambda}^{(v)} \hat{\phi}_{\lambda}^{(v)} = \hat{\gamma}_{\lambda}^{(v)} \tag{8}$$

with for all $i, j \in S\langle 0, \lambda]$,

$$\widehat{\Gamma}_{\lambda}^{(v)}(j, i) = \frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} X_{t-i} X_{t-v-j}$$

and

$$\hat{\gamma}_{\lambda}^{(v)}(j) = \frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} X_t X_{t-v-j}.$$

For the convenience of the reader, we list below assumptions that will be considered.

Assumption A1. $(X_t)_{t \in \mathbb{Z}^d}$ is a spatial ARMA(p, q) with $p \neq 0$.

Assumption A2. $(X_t)_{\in \mathbb{Z}^d}$ is causal.

Assumption A3. $(\epsilon_t)_{t \in \mathbb{Z}^d}$ is a family of i.i.d centered random variables with variance $\sigma^2 > 0$.

Assumption A4. $\mathbb{E}(\epsilon_t^4) = \eta\sigma^4$ with η being some positive constant.

3. Some Asymptotic Results for Linear Random Fields

We consider now two linear random fields $(X_t)_{t\in\mathbb{Z}^d}$, $(Y_t)_{t\in\mathbb{Z}^d}$ under the form

$$X_t = \sum_{j\in\mathbb{Z}^d} \psi_j \epsilon_{t-j}, \quad Y_t = \sum_{j\in\mathbb{Z}^d} \theta_j \epsilon_{t-j}, \tag{9}$$

with the crosscovariance function $\gamma_{xy}(\cdot)$. We first establish below the consistency of their sample crosscovariance function $\hat{\gamma}_{xy}(\cdot)$ as defined in (2).

Proposition 3.1. *Let $(X_t)_{t\in\mathbb{Z}^d}$ and $(Y_t)_{t\in\mathbb{Z}^d}$ be two linear random fields as defined in* (9). *Under Assumptions* A3 *and* A4, *for all $h \in \mathbb{Z}^d$,*

$$\hat{\gamma}_{xy}(h) \xrightarrow[N\to\infty]{\mathbb{P}} \gamma_{xy}(h),$$

where $\xrightarrow[N\to\infty]{\mathbb{P}}$ means the convergence in probability when N tends to infinity.

Proof. We first note that

$$\gamma_{xy}(h) = \sigma^2 \sum_{i\in\mathbb{Z}^d} \psi_i \theta_{i+h}$$

and

$$\hat{\gamma}_{xy}(h) = \sum_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \psi_i \theta_j \frac{1}{N_h} \sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}} \epsilon_{t-i}\epsilon_{t+h-j}.$$

For $i = j - h$, from the weak law of large numbers

$$\frac{1}{N_h} \sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}} \epsilon_{t-i}^2 \xrightarrow[N\to\infty]{\mathbb{P}} \sigma^2. \tag{10}$$

Now, let us prove that for $i \neq j - h$,

$$\frac{1}{N_h} \sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}} \epsilon_{t-i}\epsilon_{t+h-j} \xrightarrow[N\to\infty]{\mathbb{P}} 0. \tag{11}$$

We have

$$\mathbb{E}\left(\frac{1}{N_h} \sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}} \epsilon_{t-i}\epsilon_{t+h-j}\right)^2 = \frac{1}{N_h^2} \sum_{\substack{t_1\in S[\mathbf{1},\mathbf{N}]\\ t_1+h\in S[\mathbf{1},\mathbf{N}]}} \sum_{\substack{t_2\in S[\mathbf{1},\mathbf{N}]\\ t_2+h\in S[\mathbf{1},\mathbf{N}]}} \mathbb{E}(\epsilon_{t_1-i}\epsilon_{t_1+h-j}\epsilon_{t_2-i}\epsilon_{t_2+h-j}).$$

If $t_1 \neq t_2$ and $t_1 - i \neq t_2 + h - j$, $\mathbb{E}(\epsilon_{t_1-i}\epsilon_{t_1+h-j}\epsilon_{t_2-i}\epsilon_{t_2+h-j}) = 0$ by the independence property. If $t_1 \neq t_2$ and $t_1 - i = t_2 + h - j$, $\mathbb{E}(\epsilon_{t_1-i}\epsilon_{t_1+h-j}\epsilon_{t_2-i}\epsilon_{t_2+h-j}) = \mathbb{E}(\epsilon_{t_1-i}^2)\mathbb{E}(\epsilon_{t_2-i})\mathbb{E}(\epsilon_{t_1+h-j}) = 0$ again by the independence property.

Thus,

$$\mathbb{E}\left(\frac{1}{N_h}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}}\epsilon_{t-i}\epsilon_{t+h-j}\right)^2 = \frac{1}{N_h^2}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}}\mathbb{E}(\epsilon_{t-i}^2\epsilon_{t+h-j}^2).$$

Since $i \neq j - h$, we have

$$\mathbb{E}\left(\frac{1}{N_h}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}}\epsilon_{t-i}\epsilon_{t+h-j}\right)^2 = \frac{\sigma^4}{N_h}.$$

The right-hand side of the above equality converges to 0 as N tends to ∞ so that (11) follows from Chebyshev's inequality.

Let us denote for $K \in \mathbb{N}$,

$$Z_{N,K} = \sum_{i\in S[-\mathbf{K},\mathbf{K}]}\sum_{j\in S[-\mathbf{K},\mathbf{K}]}\psi_i\theta_j\frac{1}{N_h}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}}\epsilon_{t-i}\epsilon_{t+h-j}$$

and

$$Z_N = \sum_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\psi_i\theta_j\frac{1}{N_h}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t+h\in S[\mathbf{1},\mathbf{N}]}}\epsilon_{t-i}\epsilon_{t+h-j}.$$

From (10) and (11), we deduce that

$$Z_{N,K} \xrightarrow[N\to\infty]{\mathbb{P}} Z_K = \sum_{\substack{i\in S[-\mathbf{K},\mathbf{K}]\\ i+h\in S[-\mathbf{K},\mathbf{K}]}}\psi_i\theta_{i+h}\sigma^2.$$

Moreover, Z_K converges to $\gamma_{xy}(h)$ as K tends to ∞. Following Brockwell and Davis (1987, Proposition 6.3.9), it suffices then to show that

$$\lim_{K\to\infty}\limsup_{N\to\infty}\mathbb{E}|Z_N - Z_{N,K}| = 0.$$

But, this is immediately deduced from the absolute summability of $\sum\psi_j$ and $\sum\theta_j$. $\square$

Remark 3.1. Tjøstheim (1983) has shown the consistency of $\hat{\gamma}(\cdot)$ under martingale-difference assumptions for linear random fields having unilateral expansions like (4). It could be shown that Proposition 3.1 also holds under martingale-difference assumptions as in Tjøstheim (1983) by using the same proof with only some slight modifications.

Choi (2000) proved consistency for various estimators of the autocovariance function of a spatial linear random field. This result is a special case of Proposition 3.1 when $X_t = Y_t$.

We consider again the two linear random fields $(X_t)_{t\in\mathbb{Z}^d}$ and $(Y_t)_{t\in\mathbb{Z}^d}$ and denote by $\gamma_x(h)$ and $\gamma_y(h)$ the covariance functions of (Y_t) and (X_t), respectively.

Proposition 3.2. *Let* $(X_t)_{t\in\mathbb{Z}^d}$ *and* $(Y_t)_{t\in\mathbb{Z}^d}$ *be two linear random fields as in* (9). *If Assumptions* A3 *and* A4 *hold, then for* $p \in S\langle 0, \infty]$ *and* $q \in S[0, \infty]$,

$$\lim_{N\to\infty} N^d \mathrm{Cov}\left(\frac{1}{N^d}\sum_{t\in S[1,\mathbf{N}]} Y_t X_{t-q-u}, \frac{1}{N^d}\sum_{t\in S[1,\mathbf{N}]} Y_t X_{t-q-v}\right) = V(u, v) \quad \forall u, v \in S\langle 0, p]$$

with

$$\begin{aligned} V(u, v) &= (\eta - 3)\gamma_{yx}(-q-u)\gamma_{yx}(-q-v) \\ &\quad + \sum_{h\in\mathbb{Z}^d} \gamma_y(h)\gamma_x(h-u+v) + \gamma_{yx}(-h-q-v)\gamma_{yx}(h-q-u). \end{aligned}$$

Proof. Observe first that for $i, j, k, l \in \mathbb{Z}^d$

$$\mathbb{E}(\epsilon_i\epsilon_j\epsilon_k\epsilon_l) = \begin{cases} \eta\sigma^4 & \text{if } i = j = k = l, \\ \sigma^4 & \text{if } i = j \neq k = l, \\ \sigma^4 & \text{if } i = k \neq j = l, \\ \sigma^4 & \text{if } i = l \neq j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Now for $h \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathbb{E}(Y_t X_{t-q-u} Y_{t-h-q-u} X_{t-h-2q-u-v}) \\ &\quad = \sum_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \theta_i\psi_{j-q-u}\theta_{k-q-u-h}\psi_{l-2q-u-v-h}\mathbb{E}(\epsilon_{t-i}\epsilon_{t-j}\epsilon_{t-k}\epsilon_{t-l}). \end{aligned}$$

From above,

$$\begin{aligned} &\mathbb{E}(Y_t X_{t-q-u} Y_{t-h-q-u} X_{t-h-2q-u-v}) \\ &\quad = (\eta-3)\sigma^4 \sum_{i\in\mathbb{Z}^d} \theta_i\psi_{i-q-u}\theta_{i-q-u-h}\psi_{i-2q-u-v-h} + \gamma_{yx}(-q-u)\gamma_{yx}(-q-v) \\ &\qquad + \gamma_y(q+u+h)\gamma_x(q+v+h) + \gamma_{yx}(-2q-u-v-h)\gamma_{yx}(h). \end{aligned}$$

It follows that

$$\begin{aligned} &\mathrm{Cov}\left(\frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-u}, \frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-v}\right) \\ &\quad = \frac{1}{N^{2d}}\sum_{s\in S[\mathbf{1},\mathbf{N}]}\sum_{t\in S[\mathbf{1},\mathbf{N}]} \mathbb{E}(Y_t X_{t-q-u} Y_s X_{s-q-v}) - \gamma_{yx}(-q-u)\gamma_{yx}(-q-v) \\ &\quad = \frac{1}{N^{2d}}\sum_{s\in S[\mathbf{1},\mathbf{N}]}\sum_{t\in S[\mathbf{1},\mathbf{N}]} \Bigg[(\eta-3)\sigma^4 \sum_{i\in\mathbb{Z}^d} \theta_i\psi_{i-q-u}\theta_{i+(s-t)}\psi_{i-q+(s-t)-v} \\ &\qquad + \gamma_y(t-s)\gamma_x(t-s-u+v) + \gamma_{yx}(s-t-q-v)\gamma_{yx}(t-s-q-u)\Bigg]. \end{aligned}$$

Setting $h = t - s$ and permutating the two summations yield

$$N^d \operatorname{Cov}\left(\frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-u}, \frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-v}\right)$$

$$= \frac{1}{N^{2d}} \sum_{h\in S[-(\mathbf{N}-\mathbf{1}),\mathbf{N}-\mathbf{1}]} \prod_{i=1\ldots d}\left(1-\frac{|h_i|}{N}\right)\Bigg[(\eta-3)\sigma^4 \sum_{i\in\mathbb{Z}^d} \theta_i \psi_{i-q-u}\theta_{i-h}\psi_{i-q-h-v}$$

$$+\gamma_y(h)\gamma_x(h-u+v)+\gamma_{yx}(-h-q-v)\gamma_{yx}(h-q-u)\Bigg].$$

As a consequence,

$$\lim_{N\to\infty} N^d \operatorname{Cov}\left(\frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-u}, \frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_t X_{t-q-v}\right)$$

$$= (\eta-3)\gamma_{yx}(-q-u)\gamma_{yx}(-q-v) + \sum_{h\in\mathbb{Z}^d}\gamma_y(h)\gamma_x(h-u+v)$$

$$+\gamma_{yx}(-h-q-v)\gamma_{yx}(h-q-u),$$

which is exactly $V(u, v)$. □

Remark 3.2. Proposition 3.2 extends to random fields the well-known results for two time series. More precisely, for the special case $d = 1$ and $X_t = Y_t$, Proposition 3.2 corresponds to Proposition 7.3.1 in Brockwell and Davis (1987). Furthermore, for random fields, Lemma 8 in Choi (2000) is a particular case of Proposition 3.2 when $Y_t = \epsilon_t$.

4. Estimation and Identification for a Spatial ARMA

We are now interested in asymptotic behavior of estimators for the autoregressive parameters of a spatial ARMA(p, q) random field $(X_t)_{t\in\mathbb{Z}^d}$ with $p \neq 0$. First the consistency of the autoregressive coefficients estimators is stated in the following proposition.

Proposition 4.1. *If Assumptions* A1, A2, A3, *and* A4 *hold, then for all* $i, j \in S\langle 0, \lambda]$,

$$\widehat{\Gamma}_\lambda^{(v)}(j, i) \xrightarrow[N\to\infty]{\mathbb{P}} \gamma(v + j - i)$$

and

$$\hat{\gamma}_\lambda^{(v)}(j) \xrightarrow[N\to\infty]{\mathbb{P}} \gamma(v + j).$$

Proof. For $i \in S[0, \lambda]$, $j \in S\langle 0, \lambda]$, we have

$$\frac{1}{N^d}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t-\tau\in S[\mathbf{1},\mathbf{N}]}} X_{t-i}X_{t-v-j} = \frac{1}{N^d}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t-i\in S[\mathbf{1},\mathbf{N}]\\ t-v-j\in S[\mathbf{1},\mathbf{N}]}} X_{t-i}X_{t-v-j} - R_N^1,$$

with

$$R_N^1 = \frac{1}{N^d} \sum_{t \in I_N^1} X_{t-i} X_{t-v-j}$$

and

$$I_N^1 = \{t \in S[\mathbf{1}, \mathbf{N}], t - i \in S[\mathbf{1}, \mathbf{N}], t - v - j \in S[\mathbf{1}, \mathbf{N}], t - \tau \notin S[\mathbf{1}, \mathbf{N}]\}.$$

But,

$$\# I_N^1 \leq \#\{t \in S[\mathbf{1}, \mathbf{N}], t - \tau \notin S[\mathbf{1}, \mathbf{N}]\} \leq \sum_i \tau_i N^{d-1},$$

where $\# E$ denotes the cardinal of a finite subset E of $\mathbb{Z}^d$. Therefore,

$$\mathbb{E}|R_N^1| \leq \gamma_x(0) \frac{\sum_i \tau_i}{N}$$

which implies that $R_N^1 = o_{\mathbb{P}}(1)$.

Furthermore,

$$\frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-i \in S[\mathbf{1},\mathbf{N}] \\ t-v-j \in S[\mathbf{1},\mathbf{N}]}} X_{t-i} X_{t-v-j} = \frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t+(i-v-j) \in S[\mathbf{1},\mathbf{N}]}} X_t X_{t+(i-v-j)} - R_N^2$$

with

$$R_N^2 = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{t \in I_N^2} X_t X_{t+(i-v-j)} & \text{if } i \neq 0 \end{cases}$$

where $I_N^2 = \{t \in S[\mathbf{1}, \mathbf{N}], t + (i - v - j) \in S[\mathbf{1}, \mathbf{N}], t + i \notin S[\mathbf{1}, \mathbf{N}]\}$.

As for R_N^1, we show that $R_N^2 = o_{\mathbb{P}}(1)$.

Consequently, we have

$$\frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} X_{t-i} X_{t-v-j} = \frac{N_{i-v-j}}{N^d} \frac{1}{N_{i-v-j}} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t+(i-v-j) \in S[\mathbf{1},\mathbf{N}]}} X_t X_{t+(i-v-j)} + o_{\mathbb{P}}(1)$$

which converges in probability to $\gamma(i - v - j)$ by Proposition 3.1. □

From Eqs. (6) and (8), the following theorem is established.

Theorem 4.1. *If Assumptions* A1, A2, A3, *and* A4 *hold and if for some* $\lambda \in S\langle 0, \infty]$ *and* $v \in S[0, \infty]$, $\Gamma_\lambda^{(v)}$ *is invertible , then*

$$\hat{\phi}_\lambda^{(v)} \xrightarrow[N \to \infty]{\mathbb{P}} \phi_\lambda^{(v)}.$$

Remark 4.1. For causal AR(p) models, Tjøstheim (1983) has proved the strong consistency of the LS estimator of the vector of autoregressive coefficients under the assumption that the ϵ_t are strong martingale-differences and strictly stationary. Since, for an AR(p) model, $\hat{\phi}_\lambda^{(v)}$ coincides with the LS estimator when $\lambda = p$ and $v = 0$, in the case of i.i.d. innovations, Theorem 4.1 extends in fact Tjøstheim's result to ARMA(p, q) random fields.

In order to study the asymptotic normality of the estimator $\hat{\phi}_\lambda^{(v)}$ of the vector of coefficients $\phi_\lambda^{(v)}$, we consider the $S\langle 0, \lambda]$-indexed vector $\widehat{\Gamma}_\lambda^{(v)}(\hat{\phi}_\lambda^{(v)} - \phi_\lambda^{(v)})$ whose jth component equals

$$\frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} Y_{\lambda,t}^{(v)} X_{t-v-j}.$$

First, we consider truncated random fields and sums over $S[\mathbf{1}, \mathbf{N}]$. More precisely, for some $K \in \mathbb{N}$, we study the $S\langle 0, \lambda]$-indexed vector with components

$$\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v),K} X_{t-v-j}^K, \tag{12}$$

where

$$X_t^K = \sum_{j \in S[0,\mathbf{K}]} \psi_j \epsilon_{t-j}$$

and

$$Y_{\lambda,t}^{(v),K} = X_t^K - \sum_{j \in S\langle 0,\lambda]} \phi_{\lambda,j}^{(v)} X_{t-j}^K$$

which can be rewritten as follows

$$Y_{\lambda,t}^{(v),K} = \sum_{j \in S[0,\mathbf{K}+\lambda]} \beta_j^K \epsilon_{t-j}$$

while using the expansion of X_t^K.

According to the Cramér–Wold device, the asymptotic normality of the vector whose jth component is defined in (12) amounts to that of the term

$$W_{N,K} = \frac{1}{N^d} \sum_{t \in S[1,\mathbf{N}]} Y_{\lambda,t}^{(v),K} \sum_{j \in S\langle 0,\lambda]} r_j X_{t-v-j}^K$$

where $(r_j)_{j \in S\langle 0,\lambda]}$ are arbitrary vectors of reals.

As in Choi (2000), the tool we use to establish the asymptotic normality of the quantities $W_{N,K}$ is based on m-dependent random fields whose definition is recalled below.

Definition 4.1. For $m \in S[0, \infty]$, a random field $(Z_t)_{t \in \mathbb{Z}^d}$ is m-dependent if it is stationary and if for any $s, t \in \mathbb{Z}^d$, Z_s and Z_t are independent when $|t_i - s_i| > m_i$ for at least one $i = 1 \ldots d$.

For such random fields, we quote from Choi (2000) the following central limit theorem.

Theorem 4.2. *Let* $(Z_t)_{t\in\mathbb{Z}}$ *be a stationary m-dependent random field with mean zero and autocovariance function* $\gamma(\cdot)$. *If* $v_m = \sum_{h\in S[-m,m]}\gamma(h) < \infty$, *then*

$$\lim_{N\to\infty} N^d \mathrm{Var}\left(\frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Z_t\right) = v_m$$

and

$$\frac{1}{N^{d/2}}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Z_t \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, v_m),$$

where $\xrightarrow[N\to\infty]{\mathscr{L}}$ *means the convergence in distribution as N tends to infinity.*

The following lemma shows that $W_{N,K}$ is a sum of $(N+\tau)$-dependent random variables.

Lemma 4.1. *If* $(\epsilon_t)_t$ *satisfies Assumption* A3, *then* $(Y_{\lambda,t}^{(v),K}\sum_{j\in S\langle 0,\lambda]} r_j X_{t-v-j}^K)_{t\in\mathbb{Z}^d}$ *is a* $(K+\tau)$*-dependent random field.*

Proof. Let $Z_t = \left(Y_{\lambda,t}^{(v),K}X_{t-v-j}^K\right)_{j\in S\langle 0,\lambda]}$. Then $\left(Y_{\lambda,t}^{(v),K}\sum_{j\in S\langle 0,\lambda]} r_j X_{t-v-j}^K\right)_{t\in\mathbb{Z}^d}$ is $(K+\tau)$-dependent as soon as $(Z_t)_{t\in\mathbb{Z}^d}$ is.

Observe that (Z_t) is strictly stationary because the ϵ_t are i.i.d.

For $h\in\mathbb{Z}^d$ such that $h_l \neq \tau_l + K$, for example $h_l > \tau_l + K$, we show that Z_t and Z_{t+h} are independent. For this, observe that

$$Z_t = \left(\sum_{i\in S[0,\mathbf{K}+\lambda]}\beta_i^K\epsilon_{t-i}\right)\left(\sum_{k\in S[j+v,\mathbf{K}+j+v]}\psi_{k-v-j}\epsilon_{t-k}\right),$$

so that Z_t is a function of the ϵ_{t-i} for $i_l \geq 0$. In the same way,

$$Z_{t+h} = \left(\sum_{i\in S[-h,\mathbf{K}+\lambda-h]}\beta_{i+h}^K\epsilon_{t-i}\right)\left(\sum_{k\in S[v+j-h,\mathbf{K}+v+j-h]}\psi_{k-v-j+h}\epsilon_{t-k}\right)$$

only depends on the ϵ_i for $i_l < 0$.

The independence of Z_t and Z_{t+h} follows from that of the ϵ_t's. Hence (Z_t) is $(\mathbf{K}+\tau)$-dependent. □

Let us denote by $\gamma_{xy}^K(h)$ the crosscovariance function $\mathbb{E}\left(Y_t^K X_{t+h}^K\right)$ and by $\gamma_x^K(h)$, $\gamma_y^K(h)$ the covariances of X_t^K and $Y_{\lambda,t}^{(v)K}$, respectively. The following lemma gives the asymptotic normality of the truncated term $W_{N,K}$ as N tends to infinity.

Lemma 4.2. *Assume that* $(\epsilon_t)_t$ *satisfies Assumptions* A3 *and* A4. *Then*

$$N^{d/2}(W_{N,K} - \mathbb{E}(W_{N,K})) \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, v_K),$$

where $v_K = r'V_K r$ with A' denoting the transpose of a matrix A and for $u, v \in S\langle 0, \lambda]$,

$$V_K(u, v) = (\eta - 3)\gamma_{yx}^K(-\nu - u)\gamma_{yx}^K(-\nu - v) \\ + \sum_{h\in\mathbb{Z}^d} \gamma_y^K(h)\gamma_x^K(h - u + v) + \gamma_{yx}^K(-h - \nu - v)\gamma_{yx}^K(h - \nu - u).$$

Proof. From Theorem 4.2, it follows that

$$N^{d/2}(W_{N,K} - \mathbb{E}(W_{N,K})) \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, v_K),$$

where

$$\begin{aligned} v_K &= \lim_{N\to\infty} N^d \mathrm{Var}(W_{N,K}) \\ &= \lim_{N\to\infty} r' N^d \mathrm{Var}\left[\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(\nu)K} X_{t-\lambda-j}^K\right)_{j\in S\langle 0,\lambda]}\right] r \\ &= r' V_K r. \end{aligned}$$

From Proposition 3.2,

$$V_K(u, v) = (\eta - 3)\gamma_{xy}^K(-\nu - u)\gamma_{xy}^K(-\nu - v) \\ + \sum_{h\in\mathbb{Z}^d} \gamma_y^K(h)\gamma_x^K(h - u + v)\gamma_{xy}^K(-h - \nu - v)\gamma_{xy}^K(h - \nu - u)$$

for $u, v \in S\langle 0, \lambda]$. □

The following lemma establishes the asymptotic normality of the quantities

$$W_N = \frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(\nu)} \sum_{j\in\langle 0,\lambda]} r_j X_{t-\nu-j}$$

which correspond to the original untruncated terms.

Lemma 4.3. *If Assumptions* A1, A2, A3, *and* A4 *hold, then*

$$N^{d/2} W_N \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, r'Vr),$$

where for $u, v \in S\langle 0, \lambda]$

$$V(u, v) = \sum_{h\in\mathbb{Z}^d} \gamma_y(h)\gamma_x(h - u + v) + \gamma_{yx}(-h - \nu - v)\gamma_{yx}(h - \nu - u).$$

Proof. From Lemma 4.2,

$$N^{d/2}(W_{N,K} - \mathbb{E}(W_{N,K})) \xrightarrow[N\to\infty]{\mathscr{L}} W_K = \mathcal{N}(0, v_K)$$

with $v_K = r'V_K r$. But for $u, v \in S\langle 0, \lambda]$,

$$\lim_{K\to\infty} V_K(u, v) = (\eta - 3)\gamma_{yx}(-v - u)\gamma_{yx}(-v - v) + \sum_{h\in\mathbb{Z}^d} \gamma_y(h)\gamma_x(h - u + v) + \gamma_{yx}(-h - v - v)\gamma_{yx}(h - v - u).$$

Moreover, from Eqs. (5), the first term in the asymptotic variance equals 0. As a consequence, $W_K \xrightarrow[K\to\infty]{\mathscr{L}} \mathcal{N}(0, r'Vr)$.

Now to prove the lemma, we use again Proposition 6.3.9 in Brockwell and Davis (1987) and show that for all $\epsilon > 0$,

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbb{P}(N^{d/2}|W_N - (W_{N,K} - \mathbb{E}(W_{N,K})| > \epsilon) = 0.$$

But according to Chebyshev's inequality, it amounts to prove that for every $j \in S\langle 0, \lambda]$,

$$\lim_{K\to\infty} \limsup_{N\to\infty} N^d \mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} \left(Y_{\lambda,t}^{(v)} X_{t-v-j} - Y_{\lambda,t}^{(v)K} X_{t-v-j}^K\right)\right) = 0.$$

For this, observe that

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} \left(Y_{\lambda,t}^{(v)} X_{t-v-j} - Y_{\lambda,t}^{(v)K} X_{t-v-j}^K\right)\right)\\
&\quad = \mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}\right) + \mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)K} X_{t-v-j}^K\right)\\
&\qquad - 2\mathrm{Cov}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}, \frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)K} X_{t-v-j}^K\right).
\end{aligned}$$

By Proposition 3.2,

$$\begin{aligned}
&\lim_{N\to\infty} N^d \mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}\right)\\
&\quad = (\eta - 3)\gamma_{yx}(-v - j)^2 + \sum_{h\in\mathbb{Z}^d} \gamma_y(h)\gamma_x(h) + \gamma_{yx}(-h - v - j)\gamma_{yx}(h - v - j).
\end{aligned}$$

Since $\gamma_{yx}(-v - j) = 0$, the last term equals $V(j, j)$.

Again by Proposition 3.2,

$$\begin{aligned}
&\lim_{N\to\infty} N^d \mathrm{Var}\left(\frac{1}{N^d} \sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)K} X_{t-v-j}^K\right)\\
&\quad = (\eta - 3)\gamma_{yx}^K(-v - j)^2 + \sum_{h\in\mathbb{Z}^d} \gamma_y^K(h)\gamma_x^K(h) + \gamma_{yx}^K(-h - v - j)\gamma_{yx}^K(h - v - j).
\end{aligned}$$

But,

$$\gamma_x^K(h) = \sum_{j \in S[0, K-h]} \psi_j \psi_{j+h} \sigma^2,$$

which converges to $\gamma_x(h)$ when K tends to infinity.

Furthermore,

$$\gamma_y^K(h) = \sum_{j \in S\langle 0, \lambda]} \sum_{k \in S\langle 0, \lambda]} \phi_{\lambda,j}^{(v)} \phi_{\lambda,k}^{(v)} \gamma_x^K(h - k + j).$$

Thus, $\gamma_y^K(h)$ converges to $\gamma_y(h)$ when K tends to infinity.

In the same way, it can be shown that $\gamma_{yx}^K(h)$ converges to $\gamma_{yx}(h)$ so that finally,

$$\lim_{K\to\infty} \lim_{N\to\infty} N^d \mathrm{Var}\left(\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)K} X_{t-v-j}^K \right) = V(j, j).$$

Using the same arguments as in Proposition 3.2,

$$\begin{aligned}
&\lim_{N\to\infty} N^{d/2} \mathrm{Cov}\left(\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}, \frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} \left(Y_{\lambda,t}^{(v)K} X_{t-v-j}^K \right) \right) \\
&\quad = (\eta - 3)\gamma_{yx}(-v-j)\gamma_{yx^K}(-v-j) + \sum_{h \in \mathbb{Z}^d} \gamma_{yy^K}(h)\gamma_{xx^K}(h) \\
&\qquad + \gamma_{yx^K}(-h-v-j)\gamma_{y^K x}(h-v-j).
\end{aligned}$$

Observing that $\gamma_{yx}(-v-j) = 0$ and taking the limit when K tends to infinity yields

$$\lim_{K\to\infty} \lim_{N\to\infty} N^{d/2} \mathrm{Cov}\left(\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}, \frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} \left(Y_{\lambda,t}^{(v)K} X_{t-v-j}^K \right) \right) = V(j, j). \qquad \square$$

Now, from Lemma 4.3,

$$r' N^{d/2} \left(\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j} \right)_{j \in S\langle 0, \lambda]} \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, r'Vr)$$

for all arbitrary vector of reals $r = (r_j)_{j \in S\langle 0, \lambda]}$. Hence, from the Cramér–Wold device, Proposition 4.2 follows.

Proposition 4.2. *If Assumptions* A1, A2, A3, *and* A4 *hold, then*

$$N^{d/2} \left(\frac{1}{N^d} \sum_{t \in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j} \right)_{j \in S\langle 0, \lambda]} \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, V).$$

The next proposition states the asymptotic normality of the statistics

$$\frac{1}{N^d} \sum_{\substack{t \in S[\mathbf{1},\mathbf{N}] \\ t-\tau \in S[\mathbf{1},\mathbf{N}]}} Y_{\lambda,t}^{(v)} X_{t-v-j}.$$

Proposition 4.3. *Under Assumptions* A1, A2, A3, A4,

$$N^{d/2}\left(\widehat{\Gamma}_{\lambda}^{(v)}\left(\hat{\phi}_{\lambda}^{(v)} - \phi_{\lambda}^{(v)}\right)\right) \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}(0, V),$$

here for $u, v \in S\langle 0, \lambda]$,

$$V(u, v) = \sum_{h\in\mathbb{Z}^d} \gamma_y(h)\gamma_x(h - u + v) + \gamma_{yx}(-h - v - v)\gamma_{yx}(h - v - u).$$

Proof. Let us prove that for all $j \in S\langle 0, \lambda]$,

$$N^{d/2}\left(\frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j} - \frac{1}{N^d}\sum_{\substack{t\in S[\mathbf{1},\mathbf{N}]\\ t-\tau\in S[\mathbf{1},\mathbf{N}]}} Y_{\lambda,t}^{(v)} X_{t-v-j}\right) = o_{\mathbb{P}}(1).$$

By the same calculations as in Proposition 3.2,

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N^{d/2}}\sum_{t\in S[\mathbf{1},\mathbf{N}]\setminus S[\mathbf{1}+\tau,\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}\right)^2\\
&= \frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]\setminus S[\mathbf{1}+\tau,\mathbf{N}]}\ \sum_{s\in S[\mathbf{1},\mathbf{N}]\setminus S[\mathbf{1}+\tau,\mathbf{N}]}\Bigg[(\eta-3)\sigma^4\sum_{i\in\mathbb{Z}^d}\theta_i\psi_{i-v-j}\theta_{i+(s-t)}\psi_{i-v+(s-t)-j}\\
&\quad + \gamma_{yx}((s-t) - v - j)\gamma_{yx}((t-s) - v - j) + \gamma_y(t-s)\gamma_x(t-s)\Bigg].
\end{aligned}$$

Now, set $h = t - s$ and

$$T_h = (\eta-3)\sigma^4\sum_{i\in\mathbb{Z}^d}\theta_i\psi_{i-v-j}\theta_{i-h}\psi_{i-v-h-j} + \gamma_{yx}(-h - v - j)\gamma_{yx}(h - v - j) + \gamma_y(h)\gamma_x(h).$$

Since $\sum|\psi_j| < \infty$ and $\sum|\theta_j| < \infty$, it follows that $\sum|T_h| < \infty$. Hence,

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{N^{d/2}}\sum_{t\in S[\mathbf{1},\mathbf{N}]\setminus S[\mathbf{1}+\tau,\mathbf{N}]} Y_{\lambda,t}^{(v)} X_{t-v-j}\right)^2 &\le \frac{1}{N^d}\sum_{t\in S[\mathbf{1},\mathbf{N}]\setminus S[\mathbf{1}+\tau,\mathbf{N}]}\ \sum_{h\in\mathbb{Z}^d}|T_h|\\
&\le \frac{\sum_{i=1\dots d}\tau_i}{N}\sum_{h\in\mathbb{Z}^d}|T_h|.
\end{aligned}$$

Since the last term converges to 0 when N tends to infinity, the proof then ends. □

The following theorem is immediately deduced from Proposition 4.3.

Theorem 4.3. *If Assumptions* A1, A2, A3, *and* A4 *hold and if for some* $\lambda \in S\langle 0, \infty]$ *and* $v \in S[0, \infty]$, $\Gamma_{\lambda}^{(v)}$ *is invertible, then*

$$N^{d/2}(\phi_{\lambda}^{(v)} - \hat{\phi}_{\lambda}^{(v)}) \xrightarrow[N\to\infty]{\mathscr{L}} \mathcal{N}\left(0, \Sigma_{\lambda}^{(v)} V \Sigma_{\lambda}^{(v)\prime}\right)$$

with $\Sigma_{\lambda}^{(v)} = \left(\Gamma_{\lambda}^{(v)}\right)^{-1}$ *and* V *as defined in Proposition* 4.3.

Remark 4.2. The asymptotic normality for the LS estimator of the vector of autoregressive coefficients in the case of a spatial causal AR(p) model was obtained by Tjøstheim (1983). If the innovations are i.i.d., this result is a special case of Theorem 4.3, for $v = 0$ and $\lambda = p$.

5. Application—Simulation Results

We illustrate now an application of some asymptotic results obtained above to identify the following two-dimensional lattice spatial ARMA((1, 1), (1, 1)):

$$\begin{aligned} X_t - 0.8X_{(t_1-1,t_2)} + 0.56X_{(t_1-1,t_2-1)} - 0.7X_{(t_1,t_2-1)} \\ = \epsilon_t + 0.5\epsilon_{(t_1-1,t_2)} + 0.3\epsilon_{(t_1-1,t_2-1)} + 0.8\epsilon_{(t_1,t_2-1)}, \end{aligned}$$

where $(\epsilon_t)_{t\in\mathbb{Z}^2}$ is a family of i.i.d. normal random variables with mean 0 and variance 1. 500 replications from the above ARMA model over the rectangle $S[\mathbf{1}, \mathbf{500}]$ are simulated and the sample mean and the sample standard deviation of the $\hat{\phi}^{(v)}_{\lambda,\lambda}$ are computed for several values of λ and v. These results are summarized in Table 1 below where the values into brackets are the standard deviations and the symbol – denotes a value greater than 10.

The criterion we choose here for identifying spatial ARMA model is based on the estimators $(\hat{\phi}^{(v)}_{\lambda,\lambda})$ that correspond to a spatial version of the sample GPAC. Indeed, as for the one-dimensional lattice case (see Choi, 1991, 1992; Woodward and Gray, 1981), we have the following properties that could be deduced from Remark 2.2:

$$\phi^{(v)}_{p,p} = \phi_p \quad \text{for } v \geq q$$

and

$$\phi^{(q)}_{\lambda,\lambda} = 0 \quad \text{for } \lambda \geq p, \ \lambda \neq p.$$

Table 1
Sample mean and sample standard deviation of $\hat{\phi}^{(v)}_{\lambda,\lambda}$ over 500 replications

$\lambda\backslash v$	(0, 0)	(1, 0)	**(1, 1)**	(0, 1)	(2, 0)	(2, 1)	(2, 2)	(1, 2)	(0, 2)
(1, 0)	0.8839 (0.0015)	0.8001 (0.0028)	0.8000 (0.0033)	0.8817 (0.0019)	0.8001 (0.0035)	0.8000 (0.0041)	0.7999 (0.0059)	0.7999 (0.0048)	0.8816 (0.0030)
(1, 1)	−0.7503 (0.0013)	−0.6594 (0.0038)	**−0.5598** (**0.0070**)	−0.6101 (0.0031)	1.2633 (–)	**−0.7943** (7.5309)	**−0.9463** (4.8276)	**−0.7276** (2.4618)	−0.5804 (2.4876)
(0, 1)	0.8458 (0.0019)	0.8437 (0.0022)	0.6997 (0.0047)	0.6998 (0.0041)	0.8437 (0.0030)	0.6996 (0.0059)	0.6995 (0.0082)	0.6997 (0.0065)	0.6998 (0.0057)
(2, 0)	−0.3389 (0.0034)	0.0002 (0.0123)	0.0002 (0.0148)	−0.3291 (0.0040)	8.4097 (–)	−0.4678 (4.9409)	0.0807 (–)	0.0004 (0.0211)	−0.3290 (0.0060)
(2, 1)	0.2977 (0.0021)	−0.0001 (0.0074)	**−0.0003** (**0.0185**)	0.1986 (0.0050)	−0.7826 (–)	0.2643 (5.5446)	3.4385 (–)	4.1879 (–)	0.2587 (6.5056)
(2, 2)	−0.1814 (0.0022)	0.0000 (0.0065)	**0.0000** (**0.0180**)	−0.0005 (0.0051)	−1.7500 (–)	−0.0068 (0.1577)	−2.6723 (–)	−0.0063 (0.1868)	0.0083 (5.4054)
(1, 2)	0.3932 (0.0018)	0.2865 (0.0058)	**−0.0018** (**0.0179**)	−0.0001 (0.0055)	0.6953 (6.1800)	0.8073 (6.2602)	0.6075 (7.0709)	4.8234 (–)	3.3525 (–)
(0, 2)	−0.4341 (0.0032)	−0.4268 (0.0038)	0.0001 (0.0125)	0.0002 (0.0109)	−0.4268 (0.0050)	−0.0002 (0.0158)	1.0313 (–)	−0.5992 (7.6359)	1.2197 (–)

Furthermore, from Theorems 4.1 and 4.3, we have

$$\hat{\phi}^{(v)}_{\lambda,\lambda} \xrightarrow[N\to\infty]{\mathbb{P}} \phi^{(v)}_{\lambda,\lambda}$$

and

$$\hat{\phi}^{(v)}_{\lambda,\lambda} \xrightarrow[n\to\infty]{\mathscr{L}} \mathcal{N}\Big(\phi^{(v)}_{\lambda,\lambda}, (\Sigma^{(v)}_{\lambda} V \Sigma^{(v)\prime}_{\lambda})_{\lambda,\lambda}\Big).$$

Applying the properties of the $\big(\hat{\phi}^{(v)}_{\lambda,\lambda}\big)$ to the ARMA((1, 1), (1, 1)) model considered here, we have

(i) $\hat{\phi}^{(v)}_{(1,1),(1,1)} = \hat{\phi}$ for $v \geq q = (1, 1)$,

(ii) $\hat{\phi}^{((1,1))}_{\lambda,\lambda} \sim 0$ for $\lambda \geq p = (1, 1)$, $\lambda \neq p$.

Table 1 in conjunction with both the above properties clearly allows identifying the ARMA((1, 1), (1, 1)) model. Indeed, when inspecting the values of the sample standard deviation of the estimators in Table 1, first, property (i) is well revealed at the intersection of the 3rd row with the 4th column (bold characters), secondly, property (ii) is well detected at the intersection of the 4th column with the 6th, 7th, and 8th rows (bold characters).

For one replication $\{X_t, t \in S[\mathbf{1}, \mathbf{500}]\}$, the obtained estimates of the AR coefficients are (0.7989, −0.5628, 0.7038).

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